Reflectionless $\mathcal{PT}$-symmetric potentials in the one-dimensional Dirac equation

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Abstract

We study the one-dimensional Dirac equation with local $\mathcal{PT}$-symmetric potentials whose discrete eigenfunctions and continuum asymptotic eigenfunctions are eigenfunctions of the $\mathcal{PT}$ operator, too: on these conditions the bound-state spectra are real and the potentials are reflectionless and conserve unitarity in the scattering process. Absence of reflection makes it meaningful to consider also $\mathcal{PT}$-symmetric potentials that do not vanish asymptotically.

1 Introduction

Reflectionless potentials, *i.e.* potentials that are transparent to incident waves at all energies, have played a special role in quantum mechanics since the basic paper by Kay and Moses[1], who formulated the problem of constructing a plane stratified dielectric medium transparent to electromagnetic radiation in terms of a one-dimensional Schrődinger equation with a potential with preassigned bound-state spectrum that transmits without reflection continuum wave functions at all incident energies. From a mathematical point of view, the Kay-Moses method is equivalent to solving a non-linear Schrődinger equation whose potential, $V(x)$, is a quadratic function of a fixed number, n, of unknown bound-state wave functions[2]; it can also be considered as a kind of Hartree-Fock potential with n occupied states for a system of particles interacting through schematic contact interactions in one space dimension[3].

More recent approaches to reflectionless potentials in non-relativistic quantum mechanics make use, among others, of Darboux transformations[4], supersymmetric hyerarchy derivations from the trivially transparent constant potential[5] and Casimir invariants of non-compact Lie groups[6], the latter method giving rise to large families of reflectionless potentials in implicit form, in addition to explicit analytical forms derived in previous approaches.

In relativistic quantum mechanics, the Kay-Moses method has been applied to the one-dimensional Dirac equation with either scalar [7][8][9] or pseudoscalar

potentials[10], since the presence of a vector component may break the transparency of the potential at all energies (see Ref.[7] and Section 3 of the present work), with notable exceptions, one of which will be discussed in detail in Section 3. The relativistic extension of the Kay-Moses method is equivalent to the solution of an auxiliary non-linear Dirac equation.

Reflectionless potentials play an interesting role in non-Hermitian theories, too, such as quasi-Hermitian quantum mechanics[11], $\mathcal{PT}$-symmetric quantum mechanics[12][13], or pseudo-Hermitian quantum mecanics[14],[15]. As is known, if a non-Hermitian potential $V(x)$ is invariant under the product of parity $\mathcal{P}$ and time reversal $\mathcal{T}$, so that $\mathcal{PT}V(x)(\mathcal{PT})^{-1} \equiv V^*(-x) = V(x)$, and the bound-state eigenfunctions of the Schrődinger Hamiltonian $H = -\frac{1}{2m}\frac{d^2}{dx^2} + V(x)$ are eigenfunctions of $\mathcal{PT}$ (exact $\mathcal{PT}$ symmetry), the corresponding eigenvalues are real. As for the continuum of scattering states, it was proved in Ref.[16] for asymptotically vanishing potentials in the Schrődinger equation that if the asymptotic wave functions are eigenstates of $\mathcal{PT}$ (exact asymptotic $\mathcal{PT}$ symmetry), the $\mathcal{PT}$-symmetric potential is reflectionless and unitarity is conserved. In Section 2 of the present work we extend the proof to the Dirac equation with potentials that admit non-zero constant limits at $x \to \pm\infty$.

Scattering from reflectionless potentials with exact asymptotic $\mathcal{PT}$ symmetry can thus be treated by the methods of standard quantum mechanics, without the need for an equivalent Hermitian formulation, which is neither exempt from technical difficulties, nor from ambiguity of interpretation: it has been shown that the equivalent Hermitian description of scattering from strongly localized non-Hermitian potentials, a Dirac delta function with complex coupling strength in Ref.[17] and a $\mathcal{PT}$-symmetric combination of delta functions in Ref. [18], implies strongly non-local metric operators and, consequently, an apparent breaking of causality due to incoming waves in the exit channel. This seems to be the price one has to pay in order to restore unitarity in the scattering process, although a new formulation of the problem[19][20], based on the discretization of the Schrődinger equation on an infinite one-dimensional lattice, has provided examples where the metric operator in the Hermitian equivalent formulation can be chosen as a diagonal matrix, called a quasi-local operator, which prevents the appearance of incoming waves in the exit channel, at the cost of a change of scale of the probability density on the left and the right of the scattering centre.

In the present state of formulation of quasi-Hermitian theories, however, we share the opinion expressed in Ref.[18], that it makes sense to treat a non-Hermitian scattering potential as an effective one, accepting that it may well involve the loss of unitarity when attention is restricted to the system itself and not its environment, with which it can exchange probability flux (see also Refs. [21][22][23]). On the other hand, reflectionless potentials are a special class of $\mathcal{PT}$-symmetric potentials that conserve unitarity even in the standard formulation of quantum mechanics; therefore, we believe that they may deserve a study of their own, not only in the standard framework, *i.e.* with a trivial metric operator, adopted here, since it does not give rise to unphysical aspects

for an isolated system, but possibly also as a test of alternative approaches.

The main scope of the present work is to investigate the behaviour of reflectionless potentials in relativistic quantum mechanics, under different conditions of Lorentz covariance, *i.e.* when they appear as vector, scalar or pseudoscalar components in the one-dimensional Dirac equation. Section 2 describes the general formalism, examples of scalar-plus-vector potentials are worked out in Section 3, pseudoscalar potentials in Section 4 and scalar potentials in Section 5. Finally, Section 6 is dedicated to conclusions and perspectives.

2 General formalism

The time-independent Dirac equation in (1+1) dimensions with vector , scalar and pseudoscalar potentials, V, S and P, respectively, reads, in units $\hbar = c = 1$

$$\left[\alpha_x p_x + \beta\left(m + S(x)\right) + i\alpha_x \beta P\left(x\right) + V(x)\right]\Psi\left(x\right) = E\Psi\left(x\right) \ . \tag{1}$$

Here, $\Psi\left(x\right)$ is a two-dimensional spinor and $p_x \equiv -i\frac{d}{dx}$. α_x and β are two anti-commuting Hermitian traceless matrices with the property $\alpha_x^2 = \beta^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1_2$, which can be identified with two Pauli matrices.

We assume for generality's sake that S, P and V can have non-zero limits at $x = \pm\infty$: $\lim_{x\to\pm\infty} S(x) = S_\pm$, and analogous notations for P and V. In the Dirac representation[24], $\alpha_x = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\beta = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the asymptotic Dirac equation reads

$$\begin{pmatrix} m + S_\pm + V_\pm - E & -i\left(\frac{d}{dx} + P_\pm\right) \\ i\left(-\frac{d}{dx} + P_\pm\right) & -m - S_\pm + V_\pm - E \end{pmatrix} \begin{pmatrix} \psi_1\left(x\right) \\ \psi_2\left(x\right) \end{pmatrix} = 0 \ . \tag{2}$$

Let us search for a solution of the following form

$$\begin{array}{c} \psi_1\left(x\right) = A_\pm e^{ik_\pm x} + B_\pm e^{-ik_\pm x} \\ \psi_2\left(x\right) = A_\pm C_\pm e^{ik_\pm x} + B_\pm D_\pm e^{-ik_\pm x} \end{array} \ , \tag{3}$$

where $A_\pm$, $B_\pm$ $C_\pm$ and $D_\pm$ are complex numbers.

Direct substitution of formulae (3) in the asymptotic Dirac equation (2) yields

$$C_\pm = \frac{k_\pm + iP_\pm}{m + S_\pm + E - V_\pm} \ , \ D_\pm = \frac{-k_\pm + iP_\pm}{m + S_\pm + E - V_\pm}$$

and the asymptotic momenta satisfy the relation

$$k_\pm^2 = \left(E - V_\pm\right)^2 - \left(m + S_\pm\right)^2 - P_\pm^2 \ , \tag{4}$$

while $A_\pm$ and $B_\pm$ remain to be fixed on boundary conditions.

Like in our previous works[25],[26], which use the same representation of Dirac matrices, the parity operator, $\mathcal{P}$, and the time reversal operator, $\mathcal{T}$, are

$$\begin{array}{c} \mathcal{P} = P_0\beta = P_0\sigma_z \ , \\ \mathcal{T} = \mathcal{K}\beta = \mathcal{K}\sigma_z \ , \end{array}$$

where P_0 changes x into $-x$ and $\mathcal{K}$ performs complex conjugation. Their product

$$\mathcal{PT} = P_0\mathcal{K}\sigma_z^2 = P_0\mathcal{K} \tag{5}$$

is thus the same as in non-relativistic quantum mechanics[16].

If the Dirac Hamiltonian on the left-hand side of Eq. (1) is $\mathcal{PT}$-symmetric, $S_+ = S_-^*$, $V_+ = V_-^*$ and $P_+ = -P_-^*$. $\mathcal{PT}$ symmetry of the potentials thus implies $k_\pm^{2*} = k_\mp^2$, which means that either $k_-^* = k_+$, or $k_-^* = -k_+$. Using formulae (3), it is easy to show that only with the choice $k_-^* = k_+$ the ratios of transmitted waves over incident waves remain finite even if the amplitudes of asymptotic wave functions may diverge at $x = \pm\infty$. This argument does not hold for the reflection coefficient: therefore, only when reflection is identically zero it makes sense to treat $\mathcal{PT}$-symmetric potentials that do not vanish asymptotically. In turn, $k_\pm^* = k_\mp$ implies $C_\pm^* = C_\mp$ and $D_\pm^* = D_\mp$. When all the potentials vanish asymptotically, the well-known expressions for free particles are recovered: $k^2 = E^2 - m^2$, $C_\pm = \frac{k}{E+m}$, $D_\pm = -\frac{k}{E+m}$.

In general, $A_\pm$ and $B_\pm$ are linear combinations of the coefficients of asymptotic expansions of two linearly independent solutions to Eq. (1), $\Psi^{(1)}(x)$ and $\Psi^{(2)}(x)$

$$\lim_{x\to\pm\infty} \Psi^{(i)}(x) = a_{i\pm}(k_\pm)\begin{pmatrix} 1 \\ C_\pm \end{pmatrix} e^{ik_\pm x} + b_{i\pm}(k_\pm)\begin{pmatrix} 1 \\ D_\pm \end{pmatrix} e^{-ik_\pm x} \quad (i = 1, 2)$$

in the general asymptotic solution

$$\lim_{x\to\pm\infty} \Psi(x) = \alpha \lim_{x\to\pm\infty} \Psi^{(1)}(x) + \beta \lim_{x\to\pm\infty} \Psi^{(2)}(x) \ ,$$

or

$$\begin{aligned} A_\pm &= \alpha a_{1\pm}(k_\pm) + \beta a_{2\pm}(k_\pm) \ , \\ B_\pm &= \alpha b_{1\pm}(k_\pm) + \beta b_{2\pm}(k_\pm) \ . \end{aligned}$$

α and β, in turn, can be fixed by boundary conditions. If $\Psi(x)$ is a progressive wave, travelling from left to right, we must have, apart from a global normalization constant, not relevant in this context,

$$\left\{ \begin{array}{c} \lim_{x\to-\infty} \Psi(x) = \begin{pmatrix} 1 \\ C_- \end{pmatrix} e^{ik_- x} + R_{L\to R}\begin{pmatrix} 1 \\ D_- \end{pmatrix} e^{-ik_- x} \ , \\ \lim_{x\to+\infty} \Psi(x) = T_{L\to R}\begin{pmatrix} 1 \\ C_+ \end{pmatrix} e^{ik_+ x} \ , \end{array} \right.$$

where the transmission and reflection coefficients, $T_{L\to R}$ and $R_{L\to R}$, have been introduced.

Therefore

$$\left\{ \begin{array}{c} A_- = \alpha a_{1-} + \beta a_{2-} = 1 \ , \\ B_- = \alpha b_{1-} + \beta b_{2-} = R_{L\to R} \ , \\ B_+ = \alpha b_{1+} + \beta b_{2+} = 0 \ , \\ A_+ = \alpha a_{1+} + \beta a_{2+} = \ T_{L\to R} \ \ , \end{array} \right.$$

whence

$$\left\{ \begin{array}{l} T_{L\to R} = \frac{b_{1+}a_{2+} - a_{1+}b_{2+}}{b_{1+}a_{2-} - a_{1-}b_{2+}} \ , \\ R_{L\to R} = \frac{b_{1+}b_{2-} - b_{1-}b_{2+}}{b_{1+}a_{2-} - a_{1-}b_{2+}} \ . \end{array} \right. \tag{6}$$

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In the same way, if $\Psi(x)$ is a regressive wave, travelling from right to left

$$\left\{ \begin{array}{c} \lim_{x\to-\infty} \Psi(x) = T_{R\to L} \left(\begin{array}{c} 1 \\ D_- \end{array} \right) e^{-ik_- x} \ , \\ \lim_{x\to+\infty} \Psi(x) = \left(\begin{array}{c} 1 \\ D_+ \end{array} \right) e^{-ik_+ x} + R_{R\to L} \left(\begin{array}{c} 1 \\ C_+ \end{array} \right) e^{ik_+ x} \ . \end{array} \right.$$

Thus

$$\left\{ \begin{array}{c} A_- = \alpha a_{1-} + \beta a_{2-} = 0 \ , \\ B_- = \alpha b_{1-} + \beta b_{2-} = T_{R\to L} \ , \\ A_+ = \alpha a_{1+} + \beta a_{2+} = R_{R\to L} \ , \\ B_+ = \alpha b_{1+} + \beta b_{2+} = 1 \ . \end{array} \right.$$

whence

$$\left\{ \begin{array}{l} T_{R\to L} = \frac{a_{2-}b_{1-} - a_{1-}b_{2-}}{b_{1+}a_{2--} - a_{1-}b_{2+}} \ , \\ R_{R\to L} = \frac{a_{1+}a_{2-} - a_{1-}a_{2+}}{b_{1+}a_{2-} - a_{1-}b_{2+}} \ . \end{array} \right. \tag{7}$$

Not surprisingly, the transmission and reflection coefficients (6) and (7) are the same as in the non-relativistic case[16].

The Wronskian of two solutions of the Dirac equation, $\Psi^{(1)}(x) \equiv \left(\begin{array}{c} \psi_1^{(1)}(x) \\ \psi_2^{(1)}(x) \end{array} \right)$ and $\Psi^{(2)}(x) \equiv \left(\begin{array}{c} \psi_1^{(2)}(x) \\ \psi_2^{(2)}(x) \end{array} \right)$ is defined as

$$W(x) \equiv \left| \begin{array}{cc} \psi_1^{(1)}(x) & \psi_1^{(2)}(x) \\ \psi_2^{(1)}(x) & \psi_2^{(1)}(x) \end{array} \right| = \psi_1^{(1)}(x)\,\psi_2^{(1)}(x) - \psi_1^{(2)}(x)\,\psi_2^{(1)}(x) \ . \tag{8}$$

It is easy to check that $\frac{dW(x)}{dx} = 0$, *i.e.* $W(x) = const.$, by expressing the derivatives of the spinor components as linear combinations of the components themselves, as dictated by Eq. (1). If the two solutions are linearly independent, $W \neq 0$, of course.

Using definition (8) and asymptotic wave functions, $\Psi^{(i)}{}_\pm \equiv \lim_{x\to\pm\infty} \Psi^{(i)}(x)$, one easily obtains

$$\lim_{x\to\pm\infty} W(x) \equiv W_\pm = (a_{1\pm}b_{2\pm} - a_{2\pm}b_{1\pm})(D_\pm - C_\pm) \ . \tag{9}$$

Remembering expressions (6-7) of the transmission coefficients, formula (9) yields

$$\begin{array}{c} W_- = (a_{1-}b_{2+} - a_{2-}b_{1+})\, T_{R\to L}(D_- - C_-) \ , \\ W_+ = (a_{1-}b_{2+} - a_{2-}b_{1+})\, T_{L\to R}(D_+ - C_+) \ . \end{array}$$

and $W_- = W_+$ is equivalent to

$$T_{R\to L}(D_- - C_-) = T_{L\to R}(D_+ - C_+) \ ,$$

or

$$\frac{T_{L\to R}}{T_{R\to L}} = \frac{D_- - C_-}{D_+ - C_+} = \frac{D_- - C_-}{D_-^* - C_-^*} = e^{i\nu} \ , \tag{10}$$

Here, $\nu = 2\arg(D_- - C_-)$ is a real phase. When, in particular, all potentials vanish asymptotically, $C_- = -D_-$ are real numbers and the two transmission coefficients are equal.

The phase difference, ν, of the two transmission coefficients is different from zero when the imaginary components of the $\mathcal{PT}$-symmetric potentials do not vanish asymptotically and is present in non-relativistic quantum mechanics, too, as recently shown in Ref.[27] for a $\mathcal{PT}$-symmetric version of the hyperbolic Rosen-Morse potential.

It is worthwhile to point out that the formalism just developed refers to local potentials. For non-local potentials it has been shown that the ratio of the two transmission coefficients is not 1, but a complex number of unit modulus, even if the imaginary potentials vanish asymptotically, both in non-relativistic[28] and relativistic wave equations[26]. In this case the two reflection coefficients have the same phase, but different modulus and unitarity is broken.

Let us now apply the $\mathcal{PT}$ operator (5) to the general asymptotic wave functions

$$\lim_{x\to\pm\infty} \Psi(x) \equiv \Psi_\pm(x) = A_\pm \begin{pmatrix} 1 \\ C_\pm \end{pmatrix} e^{ik_\pm x} + B_\pm \begin{pmatrix} 1 \\ D_\pm \end{pmatrix} e^{-ik_\pm x} \ ,$$

or, more conveniently, to the following interpolating function, which coincides with the asymptotic wave functions at large $|x|$

$$\Psi_{int.}(x) = \frac{1}{2}(1 + sgn(x))\Psi_+(x) + \frac{1}{2}(1 - sgn(x))\Psi_-(x) \ .$$

By definition (5) one gets

$$\mathcal{PT}\Psi_{int.}(x) = \Psi^*_{int.}(-x) = \frac{1}{2}(1 - sgn(x))\Psi^*_+(-x) + \frac{1}{2}(1 + sgn(x))\Psi^*_-(-x) \ .$$

Imposing $\mathcal{PT}\ \Psi_{int.}(x) = e^{i\varphi}\Psi_{int.}(x)$, with φ a real phase, yields

$$\Psi^*_\pm(-x) = e^{i\varphi}\Psi_\mp(x) \ .$$

Remembering the behaviour of $k_\pm$, $C_\pm$ and $D_\pm$ under complex conjugation, we obtain the following constraints on $A_\pm$ and $B_\pm$

$$\begin{array}{l} A^*_\pm = e^{i\varphi}A_\mp \ , \\ B^*_\pm = e^{i\varphi}B_\mp \ . \end{array} \tag{11}$$

For a progressive wave ($A_- = 1$, $B_+ = 0$), this is equivalent to $T_{L\to R} = A_+ = e^{-i\varphi_-}$ and $R_{L\to R} = B_- = 0$, while, for a regressive wave ($\widetilde{A}_- = 0$, $\widetilde{B}_+ = 1$), one obtains $T_{R\to L} = \widetilde{B}_- = e^{-i\widetilde{\varphi}}$ and $R_{R\to L} = \widetilde{A}_+ = 0$. In other words, the potentials are reflectionless and conserve unitarity, since the transmission coefficients have unit modulus.

In the non-relativistic limit, $|C_{\pm}|$, $|D_{\pm}| \ll 1$ and the lower components of Dirac spinors are negligible with respect to the higher ones. Non-vanishing potentials at $x \to \pm\infty$ only affect asymptotic momenta $k_{\pm}$ and the preceding discussion and its conclusions remain valid, thus generalizing the case of short-range potentials treated in Ref.[16].

It is worthwhile to point out that potentials that behave asymptotically like $\mathcal{PT}$-symmetric step functions ($P_+ = -P_-^*$ and so on) may admit asymptotic wave functions that are eigenstates of $\mathcal{PT}$, unlike the step functions themselves, which are not reflectionless, because the asymptotic behaviour of wave functions is determined by the behaviour of the potentials in their whole domain.

In the following sections, we specialize the general interaction of Eq. (1) to scalar-plus-vector, pseudoscalar and scalar potentials and, for each type of potential, work out some examples in detail.

3 Scalar-plus-vector potentials

In the present section, we specialize Eq. (1) to a scalar-plus-vector potential with the same x dependence: $S(x) = c_S f(x)$, $V(x) = c_V f(x)$, with c_S and c_V real coupling constants.

$$[\alpha_x p_x + \beta m + (c_S\beta + c_V) f(x)] \Psi(x) = E\Psi(x) \ , \tag{12}$$

We find it convenient to adopt the Dirac representation $\alpha_x = \sigma_x$ and $\beta = \sigma_z$. The Dirac equation (12) satisfied by the spinor $\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ is thus written explicitly in matrix form

$$\begin{pmatrix} m - E + (c_V + c_S) f(x) & -i\frac{d}{dx} \\ -i\frac{d}{dx} & -m - E + (c_V - c_S) f(x) \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \tag{13}$$

which reduces to a system of two first-order equations for the unknown spinor components $\psi_1(x)$ and $\psi_2(x)$. In order to obtain analytic solutions, we limit ourselves to the particular cases $c_V = c_S$ and $c_V = -c_S$, which correspond to spin symmetry and pseudo-spin symmetry in three space dimensions[29]. Let us consider the case $c_V = c_S = c'$ first. It is easy to see that the two equations obtained from formula (13) reduce to the simple system

$$\left\{ \begin{array}{c} -\frac{d^2}{dx^2}\psi_1(x) + 2c'(E+m) f(x) \psi_1(x) = \left(E^2 - m^2\right) \psi_1(x) \equiv k^2 \psi_1(x) \\ \psi_2(x) = -\frac{i}{E+m}\frac{d}{dx}\psi_1(x) \end{array} \right. . \tag{14}$$

Here, the equation satisfied by $\psi_1(x)$ is Schrödinger-like, with the same $\mathcal{PT}$-symmetric form $f(x)$ as the original Dirac equation and an energy-dependent potential strength $s'(E) = 2c'(E+m)$ and $\psi_2(x)$ is obtained by deriving $\psi_1(x)$ with respect to x. On the r.h.s. of the first equation, $k^2 > 0$ for scattering states, while for bound states, $k^2 < 0$ implies an imaginary value of k, corresponding to

poles of the transmission coefficient. In the limiting case $k = 0$, both normalizable bound states and non-normalizable half-bound states[30], corresponding to transmission resonances, are possible, depending on the potentials under consideration.

When $c_V = -c_S = c''$, ψ_1 and ψ_2 exchange their role, since ψ_2 now satisfies a Schrödinger-like equation with the original $f(x)$ and the energy-dependent strength $s''(E) = 2c''(E - m)$, while ψ_1 is proportional to the space derivative of ψ_2.

$$\begin{cases} -\frac{d^2}{dx^2}\psi_2(x) + 2c''(E-m)\, f(x)\, \psi_2(x) = \left(E^2 - m^2\right)\psi_2(x) \equiv k^2 \psi_2(x) \\ \psi_1(x) = -\frac{i}{E-m}\frac{d}{dx}\psi_2(x) \end{cases} \tag{15}$$

Energy dependence of the coupling strengths in Eqs. (14-15) may affect the reflection properties of a $\mathcal{PT}$-symmetric potential. An example of this general behaviour is provided by the hyperbolic Scarf potential with integer coupling constants l and n

$$f(x) = -\frac{l^2 + n(n+1)}{2m}\frac{1}{\cosh^2 x} + \frac{il(2n+1)}{2m}\frac{\sinh x}{\cosh^2 x}\,, \tag{16}$$

which is known to be reflectionless in the Schrődinger equation[16] (note that the quoted reference uses units $2m = 1$, as is common in non-relativistic quantum mechanics). When inserted in the Dirac equation, it gives rise to an equivalent Schrődinger-like equation (14) where the potential maintains the same shape, but is no more reflectionless, because of the energy dependence of the coupling strengths.

On the contrary, if $f(x)$ exhibits an exact $\mathcal{PT}$ symmetry in the Schrödinger problem, it maintains it in the Dirac problem with the appropriate superposition of vector and scalar components, provided it is not connected with a particular value of the coupling strength, s' or s'', which becomes a function of E.

A notable example is provided by the $\mathcal{PT}$-symmetric potential

$$f(x) = \frac{1}{(x + i\epsilon)^2}\,, \tag{17}$$

where ϵ is an arbitrary real number, regularizing f at $x = 0$, which is a well-known example of reflectionless potential in the Schrödinger case[16]. Let us consider the case $c_V = c_S = c'$ first, so that the equation (14) satisfied by ψ_1 reads

$$-\frac{d^2}{dx^2}\psi_1(x) + \frac{2c'(E+m)}{(x+i\epsilon)^2}\psi_1(x) = \left(E^2 - m^2\right)\psi_1(x) \equiv k^2\psi_1(x) \tag{18}$$

for scattering states ($E^2 > m^2$). The above equation is Schrödinger-like and is quickly solved by introducing the complex variable $z = k(x + i\epsilon)$ and factorizing $\psi_1(z) = z^{1/2}\varphi(z)$: in fact, the equation satisfied by φ

$$z^2\frac{d^2}{dz^2}\varphi(z) + z\frac{d}{dz}\varphi(z) + \left[z^2 - 2c'(m+E) - \frac{1}{4}\right]\varphi(z) = 0 \tag{19}$$

is a Bessel equation of index $\nu^2 = 2c'(m+E) + \frac{1}{4}$. Note that ν is imaginary when $2c'(m+E) + \frac{1}{4} < 0$, which can happen, for instance, for positive c' and large negative E, or viceversa.

Two linearly independent solutions of Eq. (19) with asymptotic behaviour appropriate to scattering states are the Hankel functions of first and second type, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, respectively

$$\begin{aligned} \lim_{|z|\to\infty} H_\nu^{(1)}(z) &= \left(\tfrac{2}{\pi z}\right)^{1/2} \exp\left[i\left(z - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right)\right] \ , \\ \lim_{|z|\to\infty} H_\nu^{(2)}(z) &= \left(\tfrac{2}{\pi z}\right)^{1/2} \exp\left[-i\left(z - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right)\right] \ , \end{aligned} \tag{20}$$

valid for $\Re\nu > -1/2$, $|\arg z| < \pi$, this latter condition being ensured by the non-zero imaginary part of z, *i.e.* $\Im z = k\epsilon$. According to formulae (14), the corresponding linearly independent solutions of the Dirac equation are

$$\begin{aligned} \Psi^{(k)}(x) &\equiv \begin{pmatrix} \psi_1^{(k)}(x) \\ -\frac{i}{E+m}\frac{d}{dx}\psi_1^{(k)}(x) \end{pmatrix} = z^{1/2} \begin{pmatrix} H_\nu^{(k)}(z) \\ -i\lambda\left(\frac{d}{dz}H_\nu^{(k)}(z) + \frac{1}{2z}H_\nu^{(k)}(z)\right) \end{pmatrix} \\ &= z^{1/2} \begin{pmatrix} H_\nu^{(k)}(z) \\ -i\lambda\left(H_{\nu-1}^{(k)}(z) + \frac{1-2\nu}{2z}H_\nu^{(k)}(z)\right) \end{pmatrix} \ , \quad (k=1,2) \end{aligned}$$

where $\lambda \equiv \frac{k}{E+m}$. In order to obtain the final form of the r.h.s., use has been made of the relation $\frac{d}{dz}H_\nu^{(k)}(z) = H_{\nu-1}^{(k)}(z) - \frac{\nu}{z}H_\nu^{(k)}(z)$. The asymptotic behaviour of the Dirac spinors is

$$\lim_{x\to\pm\infty} \Psi^{(k)}(x) = \lim_{|z|\to\infty} z^{1/2} \begin{pmatrix} H_\nu^{(k)}(z) \\ -i\lambda H_{\nu-1}^{(k)}(z) \end{pmatrix} \ ,$$

or, more explicitly, using formulae (20)

$$\lim_{x\to\pm\infty} \Psi^{(1)}(x) = \left(\frac{2}{\pi}\right)^{1/2} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \exp\left(ikx - k\epsilon - i\frac{\pi}{2}\nu - i\frac{\pi}{4}\right) \tag{21}$$

and

$$\lim_{x\to\pm\infty} \Psi^{(2)}(x) = \left(\frac{2}{\pi}\right)^{1/2} \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} \exp\left(-ikx + k\epsilon + i\frac{\pi}{2}\nu + i\frac{\pi}{4}\right) \ . \tag{22}$$

Formulae (21-22) are particular cases of the asymptotic formulae of Section 2, whose constants now are

$$a_{1-} = a_{1+} = \left(\frac{2}{\pi}\right)^{1/2} \exp\left(-k\epsilon - i\frac{\pi}{2}\nu - i\frac{\pi}{4}\right) \ ,$$

$$b_{1-} = b_{1+} = 0 \ ,$$

$$a_{2-} = a_{2+} = 0 \ ,$$

$$b_{2-} = b_{2+} = \left(\frac{2}{\pi}\right)^{1/2} \exp\left(k\epsilon + i\frac{\pi}{2}\nu + i\frac{\pi}{4}\right) \ ,$$

so that formulae (6-7) immediately yield

$$T_{L\to R} = T_{R\to L} = 1 \ , \ R_{L\to R} = R_{R\to L} = 0 \ .$$

Of course, potential (17) does not sustain bound states with $k \neq 0$, because the transmission coefficients are independent of k and cannot have poles in k, or E. When $k = 0$, a non-trivial solution can exist at $E = m$: in this case, Eq. (18) reduces to

$$-\frac{d^2}{dx^2}\psi_1(x) + \frac{4c'm}{(x+i\epsilon)^2}\psi_1(x) = 0 \ . \tag{23}$$

The solution to Eq. (23) can be searched for in the form of a power, $(x+i\epsilon)^\gamma$, thus leading to an algebraic equation for γ

$$\gamma^2 - \gamma - 4c'm = 0 \ ,$$

whose solutions are

$$\gamma_{1,2} = \frac{1 \pm (1+16c'm)^{1/2}}{2} \ .$$

Depending on whether $1+16c'm \gtrless 0$, the two roots are either real or complex conjugate: in both cases, the general solution to Eq. (23) can be put in the form

$$\psi_1(x) = \alpha_1 (x+i\epsilon)^{\gamma_1} + \alpha_2 (x+i\epsilon)^{\gamma_2} \quad ,$$

where α_i $(i = 1,2)$ are to be fixed on boundary conditions. It is easy to understand that a normalizable solution, *i.e.* a bound state, can exist only when $1 + 16c'm > 1$, or $c' > 0$, by choosing $\alpha_1 = 0$. In this case, the solution for $\psi_1(x)$ reads

$$\psi_1(x) = \alpha_2 (x+i\epsilon)^{\frac{1-\beta}{2}} \ . \tag{24}$$

Here, $\beta = \sqrt{1+16c'm} > 1$ and α_2 can be determined by normalization of the complete Dirac spinor

$$\int\limits_{-\infty}^{+\infty} dx \begin{pmatrix} \psi_1^*(x) & \frac{i}{2m}\frac{d}{dx}\psi_1^*(x) \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ -\frac{i}{2m}\frac{d}{dx}\psi_1(x) \end{pmatrix} = 1 \ . \tag{25}$$

Both integrals in formula (25) can be computed analytically in terms of asymptotic expansions of the hypergeometric function, $F(A,B,C;z)$, since they can be reduced to the integral representation

$$\int (1+y^2)^{-a} dy = yF\left(\frac{1}{2}, a, \frac{3}{2}; -y^2\right) + const. \quad \left(\Re a > \frac{1}{2}\right)$$

yielding

$$\int_{-\infty}^{+\infty} (1+y^2)^{-a} dy \simeq \sqrt{\pi}\frac{\Gamma\left(a-\frac{1}{2}\right)}{\Gamma(a)} \ ,$$

where $\Gamma(a)$ is the Euler gamma function.

The final result for the normalization constant is

$$|\alpha_2|^2 = \frac{\epsilon^\beta}{\sqrt{\pi}\left[\epsilon^2 \frac{\Gamma\left(\frac{\beta}{2}-1\right)}{\Gamma\left(\frac{\beta-1}{2}\right)} + \frac{1}{4m^2}\left(\frac{1-\beta}{2}\right)^2 \frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}\right)}\right]} .$$

For the sake of completeness, we mention the case of the double root, $\gamma = 1/2$, of the characteristic equation, occurring when $c' = -\frac{1}{16m}$: the general solution to Eq. (23) for this case can be put in the form

$$\psi_1(x) = \alpha_1 (x+i\epsilon)^{1/2} + \alpha_2 (x+i\epsilon)^{1/2} \ln(x+i\epsilon)$$

and is not normalizable.

The case $c_V = -c_S = c''$ can be treated in a similar way starting from Eqs. (15): the solution of the Schrödinger-like equation for $\psi_2(x)$ is obtained in the same way as before and the bound state with $k=0$ now appears at $E=-m$.

4 Pseudoscalar potentials

The Dirac equation with a pseudoscalar potential, $P(x) \equiv c_P f(x)$, and c_P a real coupling constant, reads

$$[\alpha_x p_x + \beta m + i c_P \alpha_x \beta f(x)]\,\Psi(x) = E\Psi(x) \ . \tag{26}$$

In the Dirac representation, $i\alpha_x\beta = i\sigma_x\sigma_z = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. $\mathcal{PT}$ invariance of the interaction term implies $f^*(-x) = -f(x)$. After expressing Eq. (26) as a system of coupled equations in the two components, $\psi_1(x)$ and $\psi_2(x)$, of the Dirac spinor, $\Psi(x)$,

$$\begin{pmatrix} m-E & -i\frac{d}{dx} - iP(x) \\ -i\frac{d}{dx} + iP(x) & -m-E \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \tag{27}$$

it is almost immediate to derive the two decoupled Schrődinger-like equations satisfied by ψ_1 and ψ_2

$$\left(-\frac{1}{2m}\frac{d^2}{dx^2} + U_j(x)\right)\psi_j(x) = \frac{E^2-m^2}{2m}\psi_j(x) \equiv \varepsilon\psi_j(x) \ , \ (j=1,2) \tag{28}$$

where $U_1(x) \equiv \frac{1}{2m}(P^2(x) + \frac{d}{dx}P(x))$, $U_2(x) \equiv \frac{1}{2m}(P^2(x) - \frac{d}{dx}P(x))$. As already shown in Ref.[31], the two $\mathcal{PT}$-symmetric Hamiltonians

$$H_j \equiv -\frac{1}{2m}\frac{d^2}{dx^2} + U_j(x), \ (j=1,2) \tag{29}$$

constitute the Bose sector of a non-Hermitian representation of an $sl(1|1)$ superalgebra[32]. The corresponding super-Hamiltonian is

$$\mathcal{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} , \tag{30}$$

while the differential operators

$$L \equiv \frac{1}{\sqrt{2m}} \left(\frac{d}{dx} + P\left(x\right) \right) \ , \ M \equiv \frac{1}{\sqrt{2m}} \left(-\frac{d}{dx} + P\left(x\right) \right) \tag{31}$$

forming the two partner Hamiltonians $H_1 = LM$ and $H_2 = ML$, give also rise to the "supercharges"

$$Q_1 \equiv \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix} \ , \ Q_2 \equiv \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} \ , \tag{32}$$

which form the Fermi sector, since $\mathcal{H}$, Q_1 and Q_2 are closed under the following set of commutation and anticommutation relations

$$\{Q_1, Q_2\}_+ = \mathcal{H} \ , \ [Q_1, \mathcal{H}]_- = [Q_2, \mathcal{H}]_- = 0 \ .$$

It is worthwhile to point out that, differently from the Hermitian case, Q_2 is not the Hermitian adjoint of Q_1, but the following relations hold

$$M = (\mathcal{PT})L\,(\mathcal{PT})^{-1} = \mathcal{P}L^{\dagger}\mathcal{P}^{-1} \ , \ Q_2 = \mathcal{P}Q_1^{\dagger}\mathcal{P}^{-1} \ .$$

One thus speaks, in this case, of a $\mathcal{P}$-pseudo-supersymmetry[14]. Note that $P\left(x\right)$ plays the role of a superpotential. The fact that the supercharges (32) are not the Hermitian conjugates of each other gives rise to a richer structure of supersymmetric systems compared with Hermitian theories[33].

As is known, the discrete spectra and the scattering properties of the partner Hamiltonians are connected by supersymmetry, so that it is sufficient to compute bound states and scattering states of one partner only.

In particular, it is not difficult to prove the relations connecting the transmission and reflection coefficients of the two partners, in terms of the asymptotic limits of the superpotential, $P_{\pm} = \lim_{x\to\pm\infty} P\left(x\right)$ and the corresponding asymptotic momenta, $k_{\pm}^2 = E^2 - m^2 - P_{\pm}^2$: let $T^{(j)}_{L\to R}$ ($R^{(j)}_{L\to R}$) be the transmission (reflection) coefficient of a progressive wave in presence of potential $U^{(j)}$ $(j = 1, 2)$: we easily obtain

$$\begin{array}{r} R^{(1)}_{L\to R} = \frac{-ik_- + P_-}{ik_- + P_-} R^{(2)}_{L\to R} \\ T^{(1)}_{L\to R} = \frac{ik_+ + P_+}{ik_- + P_-} T^{(2)}_{L\to R} \end{array} \quad . \tag{33}$$

Similar relations connect the transmission and reflection coefficients of a regressive wave

$$\begin{array}{r} R^{(1)}_{R\to L} = \frac{ik_+ + P_+}{-ik_+ + P_+} R^{(2)}_{R\to L} \\ T^{(1)}_{R\to L} = \frac{-ik_- + P_-}{-ik_+ + P_+} T^{(2)}_{R\to L} \end{array} \tag{34}$$

Derivation of formulae (33-34) is given in Appendix , for the sake of completeness.

It is obvious that if one of the two partners is reflectionless, so is the other and, consequently, the superpotential in the Dirac equation.

In Ref.[10], the Kay-Moses method of constructing reflectionless potentials was extended to real symmetric pseudoscalar potentials in the Dirac equation. The examples worked out in the quoted reference can be made $\mathcal{PT}$-symmetric by applying an imaginary shift to the space coordinate, $x \to x + i\epsilon$. Thus, the following superpotential, with $c_P = 1$ for simplicity's sake

$$f(x) = \tanh(x + i\epsilon) + \frac{\lambda^2 - 1}{\tanh(x + i\epsilon) - \lambda \coth(\lambda(x + i\epsilon))} , \tag{35}$$

and $\lambda \geq 1$, generates the following supersymmetry partners

$$U_1(x) = \frac{1}{2m}\left[\lambda^2 - \frac{2(\lambda^2 - 1)(\lambda^2 \cosh^2(x + i\epsilon) + \sinh^2(\lambda(x + i\epsilon)))}{(\lambda \cosh(x + i\epsilon)\cosh(\lambda(x + i\epsilon)) - \sinh(x + i\epsilon)\sinh(\lambda(x + i\epsilon)))^2}\right] ,$$

and

$$U_2(x) = \frac{1}{2m}\left(\lambda^2 - \frac{2}{\cosh^2(x + i\epsilon)}\right) . \tag{36}$$

A part from the constant term $\lambda^2/(2m)$, which enters in the definition of the asymptotic momentum, $U_2(x)$ is a reflectionless Pőschl-Teller potential[16] and $U_1(x)$ is necessarily reflectionless, too. It is worthwhile to point out that our definitions of U_1 and U_2 are exchanged with respect to Ref.[10], but in agreement with Ref.[31]. Transmission and reflection coefficients for potential (36) can be immediately written down from the corresponding formulae of the more general hyperbolic Scarf potential obtained in Ref.[16], after observing that $f_\pm = \mp\lambda$ and the asymptotic momenta are $k_\pm = \sqrt{E^2 - m^2 - V_\pm^2} = \sqrt{E^2 - m^2 - \lambda^2} \equiv k$

$$\begin{aligned} R^{(2)}_{L\to R} &= R^{(2)}_{R\to L} = 0 , \\ T^{(2)}_{L\to R} &= T^{(2)}_{R\to L} = -\tfrac{1-ik}{1+ik} . \end{aligned} \tag{37}$$

A real k is a necessary condition for $\left|T^{(2)}_{L\to R}\right| = \left|T^{(2)}_{R\to L}\right| = 1$, equivalent to exact $\mathcal{PT}$ symmetry of the asymptotic wave functions.

Formulae (33-34) immediately yield the corresponding coefficients for potential U_1

$$\begin{aligned} R^{(1)}_{L\to R} &= R^{(1)}_{R\to L} = 0 , \\ T^{(1)}_{L\to R} &= T^{(1)}_{R\to L} = \tfrac{\lambda-ik}{\lambda+ik}\tfrac{1-ik}{1+ik} . \end{aligned}$$

As far as bound states are concerned, it is well known[32] that $U_2(x)$ admits only one bound state with eigenvalue $\varepsilon = \frac{\lambda^2-1}{2m}$ (or $E^2 = m^2 + \lambda^2 - 1$), and the corresponding wave function is

$$\psi_2(x) = \frac{N}{\cosh(x + i\epsilon)} , \tag{38}$$

with the constant N to be determined from normalization of the complete Dirac spinor.

Since $H_2\psi_2(x) = ML\psi_2(x) = \frac{\lambda^2-1}{2m}\psi_2(x)$, we have that $LH_2\psi_2(x) = LML\psi_2(x) = H_1L\psi_2(x) = \frac{\lambda^2-1}{2m}L\psi_2(x)$. Therefore $\psi_1(x) = c_0L\psi_2(x)$, with c_0 a normalization constant, is eigenfunction of H_1 with eigenvalue $\frac{\lambda^2-1}{2m}$ and corrresponds to the first component of the Dirac spinor. From the definition of the differential operator L and from the first equation (27) we get $c_0 = i\frac{\sqrt{2m}}{m-E}$ and

$$\begin{aligned}\psi_1(x) &= \frac{i}{m-E}\left(\frac{d}{dx}+f(x)\right)\frac{N}{\cosh(x+i\epsilon)} \\ &= \left(\frac{iN}{m-E}\right)\frac{\lambda^2-1}{\sinh(x+i\epsilon)-\lambda\cosh(x+i\epsilon)\coth(\lambda(x+i\epsilon))}\end{aligned} \tag{39}$$

with $\int\limits_{-\infty}^{+\infty}\Psi^\dagger(x)\Psi(x)\,dx = \int\limits_{-\infty}^{+\infty}\left(|\psi_1(x)|^2+|\psi_2(x)|^2\right)dx = 1$.

$\psi_1(x)$ from formula (39) has a node at $x=-i\epsilon$ and is not the ground state of H_1. Conversely, if a non-trivial normalizable solution of the equation $M\overline{\psi}_1(x)=0$ exists, then $H_1\overline{\psi}_1(x) = LM\overline{\psi}_1(x) = 0$, and $\overline{\psi}_1(x)$ is eigenstate of H_1 with eigenvalue $\varepsilon = 0$. In this case, $\overline{\psi}_1(x)$ cannot be written as $c_0L\overline{\psi}_2(x)$, with $\overline{\psi}_2(x)$ a non-trivial normalizable function, since, otherwise, we would have $ML\overline{\psi}_2(x) = H_2\overline{\psi}_2(x) = 0$ and $\overline{\psi}_2(x)$ would be an eigenstate of H_2 with eigenvalue $\varepsilon=0$, which is impossible, because $H_2\overline{\psi}_2(x)=0$ admits only the trivial solution $\overline{\psi}_2(x)=0$.

The equation

$$M\overline{\psi}_1(x) = \frac{1}{\sqrt{2m}}\left(-\frac{d}{dx}+f(x)\right)\overline{\psi}_1(x) = 0\,, \tag{40}$$

is satisfied by

$$\overline{\psi}_1(x) = \exp\left(\int^x f(x')\,dx'\right)\,.$$

Note that the condition $\lim_{x\to\pm\infty} f(x) \equiv f_\pm = \mp\lambda$, with $\lambda > 1$, yields $\lim_{x\to\pm\infty}\overline{\psi}_1(x) = 0$.

In our case

$$\int^x f(x)\,dx = \ln(\cosh x) - \ln(2\lambda\cosh x\cosh(\lambda x) - 2\sinh x\sinh(\lambda x)) + \ln\overline{N}\,,$$

so that

$$\overline{\psi}_1(x) = \frac{\overline{N}}{\lambda\cosh(\lambda(x+i\epsilon)) - \tanh(x+i\epsilon)\sinh(\lambda(x+i\epsilon))}\,, \tag{41}$$

where $\overline{N}$ is to be determined from normalization. The second component of the Dirac spinor, $\overline{\psi}_2(x)$, is solution of the equation $H_2\overline{\psi}_2(x) = ML\overline{\psi}_2(x) =$

$0 \Rightarrow L\overline{\psi}_2(x) = 0$, which admits only the trivial solution, $\overline{\psi}_2(x) = 0$. $\overline{N}$ is thus determined from the condition $\int_{-\infty}^{+\infty} \Psi^\dagger(x)\Psi(x)\,dx = \int_{-\infty}^{+\infty} \left|\overline{\psi}_1(x)\right|^2 dx = 1$.

In this case, the ground state of H_2 has the same energy, $\varepsilon = \frac{\lambda^2-1}{2m} > 0$, as the first excited state of H_1, whose ground state has $\varepsilon = 0$. This is an example of exact supersymmetry.

A second example of reflectionless pseudoscalar potential, whose bound states were already studied in Ref.[31], is

$$f(x) = n\tanh x + i\frac{l}{\cosh x}\ , \tag{42}$$

with integer constants n and l. In fact, the two supersymmetric partners from formula (29) are

$$\begin{array}{l} U_1(x) = \frac{1}{2m}\left(n^2 - \frac{n(n-1)+l^2}{\cosh^2 x} + il(2n-1)\frac{\sinh x}{\cosh^2 x}\right)\ , \\ U_2(x) = \frac{1}{2m}\left(n^2 - \frac{n(n+1)+l^2}{\cosh^2 x} + il(2n+1)\frac{\sinh x}{\cosh^2 x}\right)\ , \end{array}$$

which, a part from the constant term $\frac{n^2}{2m}$, are reflectionless potentials of hyperbolic Scarf type (16). In this case, $f_\pm = \pm n$ and $k_\pm = \sqrt{E^2 - m^2 - n^2} \equiv k$. Here again, the transmission coefficients are given by Ref.[16]

$$T^{(2)}_{L\to R} = T^{(2)}_{R\to L} = (-1)^{n+l}\frac{(n-ik)\dots(1-ik)\left(l-\frac{1}{2}-ik\right)\dots\left(\frac{1}{2}-ik\right)}{(n+ik)\dots(1+ik)\left(l-\frac{1}{2}+ik\right)\dots\left(\frac{1}{2}+ik\right)}$$

for $n > 1$ and

$$T^{(1)}_{L\to R} = T^{(1)}_{R\to L} = (-1)^{n+l-1}\frac{(n-1-ik)\dots(1-ik)\left(l-\frac{1}{2}-ik\right)\dots\left(\frac{1}{2}-ik\right)}{(n-1+ik)\dots(1+ik)\left(l-\frac{1}{2}+ik\right)\dots\left(\frac{1}{2}+ik\right)}$$

for $n > 2$.

Unitarity and asymptotic $\mathcal{PT}$ symmetry are conserved if k is real.

As for bound states, $U_2(x)$ admits n of them, all with real energies, and $U_1(x)$ has $n-1$ bound states at the same energies of those of $U_2(x)$, excepted the ground state of the latter. Here again, the pseudo-supersymmetry is exact.

5 Scalar potentials

When only a scalar potential, $S(x) = S^*(-x)$, is present, Eq. (1) simplifies to

$$[\alpha_x p_x + \beta(m + S(x))]\,\Psi(x) = E\Psi(x)\ . \tag{43}$$

In spite of its apparent greater simplicity, however, Eq. (43) is more difficult to solve than Eq. (12), including a vector potential with the same x dependence of the scalar potential and equal, or opposite coupling strength, if one adopts the Dirac representation $\alpha_x = \sigma_x$, $\beta = \sigma_z$, because the second order equation

satisfied by the first component, $\psi_1(x)$, of the Dirac spinor, $\Psi(x)$, now contains also a first order derivative, $\frac{d}{dx}\psi_1(x)$, whose x-dependent coefficient is negligible only at $x = \pm\infty$, provided $S(x)$ admits constant limits, $\lim_{x\to\pm\infty} S(x) = S_\pm$, as assumed in Section (2) in deriving the general form of asymptotic wave functions for arbitrary combinations of scalar, vector and pseudoscalar potentials.

If we are interested in exact wave functions, including those corresponding to bound states, it is more convenient to adopt a different representation of Dirac matrices, $\alpha_x = \sigma_y$, $\beta = \sigma_x$, like in Ref.[31]. The main drawback of this choice is that the kinetic term of the Dirac Hamiltonian, $\alpha_x p_x = \sigma_y p_x$, is not $\mathcal{PT}$-symmetric, but the remedy is simple, since the two representations are unitarily equivalent: the unitary transformation

$$U = e^{i\frac{\pi}{4}\sigma_z} e^{i\frac{\pi}{4}\sigma_y} = \frac{1}{2}\left[1 + i\left(\sigma_x + \sigma_y + \sigma_z\right)\right] = \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \tag{44}$$

changes the α_x and β matrices of the representation of Ref.[31] into those of the Dirac representation, since

$$\begin{aligned} U\sigma_y U^{-1} &= \sigma_x \ , \\ U\sigma_x U^{-1} &= \sigma_z \ . \end{aligned}$$

It is also of interest to determine the operator that corresponds in the present representation to to the $\mathcal{PT}$ operator (5) of the Dirac representation: let us notice that $\mathcal{PT}$ anticommutes with $i\sigma_x$, since $\mathcal{T}$ is antilinear, and that $(\mathcal{PT}i\sigma_x)^2 = 1_2$, the 2×2 identity matrix. It is not difficult to check that

$$U\mathcal{PT}i\sigma_x U^{-1} = -iU\sigma_x U^T \mathcal{PT} = \mathcal{PT} \ .$$

Therefore, $\mathcal{PT}i\sigma_x$ is obtained from $\mathcal{PT}$ by means of the similarity transformation that connects matrices in the present representation with those in the Dirac representation. Moreover, once the Dirac equation (43) has been solved in the new representation, and the spinor $\Psi(x)$ is known, the corresponding solution in the $\mathcal{PT}$-symmetric Dirac representation will be

$$\Psi_D(x) = U\Psi(x) \ . \tag{45}$$

As a consequence, if $\Psi_D(x)$ is an eigenstate of $\mathcal{PT}$, $\Psi(x)$ is an eigenstate of $\mathcal{PT}i\sigma_x$.

In the new representation the two equations satisfied by the components, $\psi_1(x)$ and $\psi_2(x)$, of $\Psi(x)$

$$\begin{bmatrix} -E & -\frac{d}{dx} + m + S(x) \\ \frac{d}{dx} + m + S(x) & -E \end{bmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ,$$

are easily decoupled as

$$\left\{ \begin{array}{c} -\frac{1}{2m}\frac{d^2}{dx^2}\psi_1(x) + \frac{1}{2m}\left((m + S(x))^2 - m^2 - \frac{d}{dx}S(x)\right)\psi_1(x) = \frac{E^2-m^2}{2m}\psi_1(x) \equiv \epsilon\psi_1(x) \\ \psi_2(x) = \frac{1}{E}\left(\frac{d}{dx} + m + S(x)\right)\psi_1(x) \end{array} \right. , \tag{46}$$

where the equation satisfied by $\psi_1(x)$ is Schrődinger-like, with an effective potential

$$U_1(x) = \frac{1}{2m}\left((m + S(x))^2 - m^2 - \frac{d}{dx}S(x)\right) . \tag{47}$$

If one derives instead the equation satisfied by $\psi_2(x)$, the result is a Schrődinger-like equation with the same effective energy $\epsilon = \frac{E^2-m^2}{2m}$ and an effective potential

$$U_2(x) = \frac{1}{2m}\left((m + S(x))^2 - m^2 + \frac{d}{dx}S(x)\right) . \tag{48}$$

We are thus led again to a pseudo-supersymmetry[31], like in the case of a pseudoscalar potential. Note that $S(x) + m \equiv m(x)$, the effective Dirac mass, now plays the role of a superpotential. The supercharges now are

$$L = \frac{1}{\sqrt{2m}}\left(-\frac{d}{dx} + m + S(x)\right) \ , \ M = \frac{1}{\sqrt{2m}}\left(\frac{d}{dx} + m + S(x)\right)$$

and the partner Hamiltonians

$$H_1 = LM = -\frac{1}{2m}\frac{d^2}{dx^2} + U_1(x) \ , \ H_2 = ML = -\frac{1}{2m}\frac{d^2}{dx^2} + U_2(x) \ .$$

In this representation, the asymptotic Dirac equation

$$\left[\begin{array}{cc} -E & -\frac{d}{dx} + m + S_\pm \\ \frac{d}{dx} + m + S_\pm & -E \end{array}\right]\left(\begin{array}{c} \psi_{1\pm}(x) \\ \psi_{2\pm}(x) \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right) ,$$

where, as before, $\lim_{x\to\pm\infty} S(x) = S_\pm$, $\lim_{x\to\pm\infty} \psi_k(x) = \psi_{k\pm}(x)$, is easily solved in the form

$$\begin{aligned} \Psi_\pm(x) &= \left(\begin{array}{c} \psi_{1\pm}(x) \\ \psi_{2\pm}(x) \end{array}\right) \\ &= \widetilde{A}_\pm \left(\begin{array}{c} 1 \\ \widetilde{C}_\pm \end{array}\right) e^{ik_\pm x} + \widetilde{B}_\pm \left(\begin{array}{c} 1 \\ \widetilde{D}_\pm \end{array}\right) e^{-ik_\pm x} , \end{aligned}$$

where $k_\pm = \left(E^2 - (m + S_\pm)^2\right)^{1/2}$ are the complex asymptotic momenta and

$$\begin{array}{l} \widetilde{C}_\pm = \frac{ik_\pm + m + S_\pm}{E} \\ \widetilde{D}_\pm = \frac{-ik_\pm + m + S_\pm}{E} \end{array} ,$$

while $\widetilde{A}_\pm$ and $\widetilde{B}_\pm$ are to be fixed on boundary conditions. Owing to the fact that $\Psi_\pm(x)$ and $\Psi_{D\pm}(x)$ are connected by Eqs. (44-45), the asymptotic expansion coefficients $\widetilde{A}_\pm$ and $\widetilde{B}_\pm$ are related by the inverse of formula (45), *i.e.* $\Psi(x) = U^\dagger \Psi_D(x)$, to the corresponding ones in the Dirac representations, $A_\pm$ and $B_\pm$, given in formulae (3) with $V_\pm = P_\pm = 0$, in the following way

$$\begin{array}{l} \widetilde{A}_\pm = \frac{A_\pm}{\sqrt{2}} e^{-i\frac{\pi}{4}} \frac{m + S_\pm + E - ik_\pm}{m + S_\pm + E} , \\ \widetilde{B}_\pm = \frac{B_\pm}{\sqrt{2}} e^{-i\frac{\pi}{4}} \frac{m + S_\pm + E + ik_\pm}{m + S_\pm + E} . \end{array} \tag{49}$$

It is worthwhile to stress once again that in the present representation of the Dirac equation $\Psi_{\pm}(x)$ is not eigenstate of $\mathcal{PT}$.

Before discussing specific examples, it is worthwhile to recall that a reflectionless potential can be obtained as a supersymmetry partner of the constant potential: in fact, putting $U_1(x) = c$ in formula (47), one obtains a Riccati equation for the superpotential $S(x)+m$, which can be solved by separation of variables, giving rise to different solutions in connection with the sign of the constant $m+2c$. The superpotential is a trigonometric function of x when $m+2c<0$ (or $c<-\frac{m}{2}$) and a hyperbolic function when $m+2c>0$; consequently, $U_2(x)$ is a trigonometric Pöschl-Teller potential in the former case and a hyperbolic Pöschl-Teller potential in the latter. In the limiting case $m+2c=0$, one obtains $S(x)+m=-\frac{1}{x+d}$, where d is an arbitrary constant, and $U_2(x)=\frac{1}{m}\frac{1}{(x+d)^2}-\frac{m}{2}$. When $d=i\epsilon$, with real ϵ, $U_2(x)$ is qualitatively similar, apart from the additive constant, to the potential studied in Section 3. Similar considerations could be made in the case of the supersymmetry involving pseudoscalar potentials presented in Section 4.

A simple example of reflectionless scalar potential is the $\mathcal{PT}$-symmetrized form of the real potential with one bound state derived in Ref.[8]

$$\begin{aligned} S(x) &= -\frac{2\kappa_B^2}{m+E_B\cosh\left(2\kappa_B\left(x+i\epsilon\right)\right)} \\ &= -\frac{\kappa_B^2}{E_B}\frac{1}{\cosh\left(\kappa_B\left(x+i\epsilon\right)-\lambda_B\right)\cosh\left(\kappa_B\left(x+i\epsilon\right)+\lambda_B\right)}\ . \end{aligned} \tag{50}$$

Here, $E_B=\frac{2m}{\sqrt{c_S^2+4}}$ and $\kappa_B=\sqrt{m^2-E_B^2}=\frac{c_S m}{\sqrt{c_S^2+4}}$ are energy and momentum of the bound state, expressed as functions of the coupling strength c_S in the auxiliary non-linear Dirac equation discussed in the above mentioned reference and $\lambda_B=\frac{1}{2}\text{arccosh}\left(\frac{m}{E_B}\right)$. The two partner potentials from formulae (47-48) can be written in the compact form

$$\begin{array}{l} U_1(x) = -\frac{\kappa_B^2}{m\cosh^2(\kappa_B(x+i\epsilon)-\lambda_B)}\ , \\ U_2(x) = -\frac{\kappa_B^2}{m\cosh^2(\kappa_B(x+i\epsilon)+\lambda_B)}\ . \end{array} \tag{51}$$

While $S(x)$ is $\mathcal{PT}$-symmetric, the partner potentials (51) are not, as expected from Eqs. (47-48). The two Schrődinger-like equations with potentials (51)

$$-\frac{1}{2m}\frac{d^2}{dx^2}\psi_i(x)+U_i(x)\psi_i(x)=\varepsilon\psi_i(x)\quad(i=1,2) \tag{52}$$

are satisfied by the bound-state wave functions

$$\begin{array}{l} \psi_1(x) = \frac{N_1}{\cosh(\kappa_B(x+i\epsilon)-\lambda_B)} \\ \psi_2(x) = \frac{N_2}{\cosh(\kappa_B(x+i\epsilon)+\lambda_B)} \end{array} \tag{53}$$

respectively, where N_1 and N_2 are normalization constants. Note that the two partners have the same discrete spectrum, *i.e.* one bound state with real energy

$\varepsilon = -\frac{\kappa_B^2}{2m}$: the pseudosupersymmetry is thus spontaneously broken[9]. The two normalization constants are related by the second Eq. (46): after replacing in it formula (50) for $S(x)$ and the first formula (53) for $\psi_1(x)$, some straightforward algebra gives $N_2 = N_1$. The corresponding Dirac spinor, $\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$, has real energy $E_B = \sqrt{m^2 - \kappa_B^2}$ and is normalized to one. Hence,

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} \Psi^\dagger(x)\,\Psi(x)\,dx = \int_{-\infty}^{+\infty} |\psi_1(x)|^2\,dx + \int_{-\infty}^{+\infty} |\psi_2(x)|^2\,dx \\ &= \frac{4}{\kappa_B}|N_1|^2 \int_{-\infty}^{+\infty} \frac{dy}{\cosh(2y) + \cos(2\kappa_B\epsilon)} = \frac{4\epsilon|N_1|^2}{\sin(\kappa_B\epsilon)\cos(\kappa_B\epsilon)} \\ &\Rightarrow \quad |N_1|^2 = \frac{\sin(\kappa_B\epsilon)\cos(\kappa_B\epsilon)}{4\epsilon}\,. \end{aligned}$$

Eqs. (52) with $\varepsilon > 0$ are satisfied by the scattering wave functions

$$\begin{aligned} \psi_1(x) &= \tfrac{e^{ik(x+i\epsilon)}}{ik+\kappa_b}\left[ik - \kappa_b \tanh(\kappa_b(x+i\epsilon) - \lambda_B)\right] \\ \psi_2(x) &= \tfrac{e^{ik(x+i\epsilon)}}{ik+\kappa_b}\left[ik - \kappa_b \tanh(\kappa_b(x+i\epsilon) + \lambda_B)\right] \end{aligned} \tag{54}$$

respectively, with $k = \sqrt{2m\varepsilon}$, if boundary conditions for incident progressive waves ($L \to R$) are imposed. Since $\lim_{x\to\pm\infty}\tanh(\kappa_b(x+i\epsilon) \pm \lambda_B) = \pm 1$, $\psi_i(x)$ does not contain a reflected component, so that $R^{(i)}_{L\to R} = 0$. It is also immediate to determine the transmission coefficient

$$T^{(i)}_{L\to R} = \frac{\lim_{x\to+\infty}\psi_i(x)}{\lim_{x\to-\infty}\psi_i(x)} = \frac{ik - \kappa_b}{ik + \kappa_b}\,. \tag{55}$$

Since $T^{(1)}_{L\to R} = T^{(2)}_{L\to R}$, formula (55) yields the transmission coefficient $T_{L\to R}$ of the Dirac spinor, which has unit modulus, as expected. The replacement $k \to -k$ changes progressive waves (54) into regressive waves, whose transmission coefficient now is

$$T^{(i)}_{R\to L} = \frac{\lim_{x\to-\infty}\psi_i(x)}{\lim_{x\to+\infty}\psi_i(x)} = \frac{-ik + \kappa_b}{-ik - \kappa_b} = T^{(i)}_{L\to R} \tag{56}$$

with $R^{(i)}_{R\to L} = R^{(i)}_{L\to R} = 0$. The equality of the transmission coefficients could have been proved also by means of formula (10), since, in the present case, $C_- = C_+ = \frac{ik+m}{E}$ and $D_- = D_+ = \frac{-ik+m}{E}$.

6 Comments and outlook

In the present work we have extended the analysis of reflectionless $\mathcal{PT}$-symmetric potentials in non-relativistic quantum mechanics presented in Ref.[16] to relativistic quantum mechanics with different forms of potentials in the Dirac equation: scalar, pseudoscalar , or a mixture of scalar and vector potentials. We

have examined the connection between reflectionlessness and exact $\mathcal{PT}$ symmetry of bound-state wave functions and asymptotic wave functions even in the case the potentials have non-zero limits at $x \to \pm\infty$, thus removing a constraint imposed in Ref.[16]. Along this line, one could study further reflectionless potentials that diverge at $x \to \pm\infty$, such as the class of real symmetric potentials $V(x) = -x^{2k+2}$ $(k = 1, 2, ...)$ discussed in Ref.[34]. A main drawback is that they are not exactly solvable in general and one has to resort to some approximate method, such as the WKB method of Ref.[34].

Reflectionless potentials are expected to play a peculiar role also in $\mathcal{PT}$-symmetric quantum mechanics in higher dimensions, very little explored up to the present time and only in non-relativistic problems. In three dimensions, for instance, only non-central potentials can exhibit non-trivial $\mathcal{PT}$ symmetry: in polar coordinates r, θ, ϕ, they satisfy the relation[35]

$$V(\mathbf{r}) \equiv V(r, \theta, \phi) = V^*(r, \pi - \theta, \phi + \pi) = V^*(-\mathbf{r}) \ .$$

General characteristics of bound states are discussed in Ref.[35] in case of exact $\mathcal{PT}$ symmetry and in Ref.[36] when the symmetry is spontaneously broken.

Scattering states of $\mathcal{PT}$-symmetric potentials in higher dimensions have not been discussed so far, but a complete analytic description should be possible if these non-central transparent potentials can be related to the euclidean group in n dimensions, $E(n)$, for $n > 3$, since it is the maximal symmetry group of the transparent null potential in n dimensions. For instance, Ref.[37] solves two classes of real transparent potentials in three dimensions that admit $E(4)$ as the potential group, in the sense that the corresponding Hamiltonians depend on the restriction of the quadratic Casimir operator to subspaces appearing in two subgroup reduction chains of $E(4)$. The possible group-theoretical treatment of complex potentials would require non-standard realizations of the non-compact Lie group involved (non-unitary representations), along the lines of Ref.[38]. Extension of this study to relativistic quantum mechanics would be possible by relating to the Casimir operator of a non-compact group the mass invariant operator of the system within the framework of a Bakamjian-Thomas realization of the generators of the Poincaré group, under which one has to ensure the invariance of the scattering matrix[39].

Finally, it is worthwhile to mention that interest in the first quantized version of the Dirac equation has been renewed in recent years by quantum simulations of the dynamics of Dirac fermions with controllable laboratory systems underlying the same mathematical models: single trapped ions are particularly suited to this purpose and experiments have been designed to simulate the one-dimensional trembling motion (Zitterbewegung[40],[41]) of a free Dirac fermion[42],[43], the overcriticality effects of a vector potential well in one dimension[42], the relativistic Landau levels generated in a constant homogeneous magnetic field[44], the pseudoscalar Dirac oscillator in two dimensions[45]. The possibility of simulating in quantum optical systems the one-dimensional Dirac dynamics in presence of non-Hermitian potentials, such as the $\mathcal{PT}$-symmetric ones considered in the present work, would be of mutual benefit to both fields of research.

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A Relations between transmission and reflection coefficients of supersymmetry partners in the pseudoscalar case

Let us consider a scattering solution of one of the two partner Hamiltonians, *e. g.* H_1, with energy $\epsilon \equiv \frac{E^2-m^2}{2m} \geq 0$

$$H_1\phi^{(1)}(x) = \epsilon\phi^{(1)}(x)$$

and a solution, $\phi^{(2)}$, of Hamiltonian H_2 with the same energy ϵ

$$H_2\phi^{(2)}(x) = \epsilon\phi^{(2)}(x) \ .$$

Since $H_1 = LM$ and $H_2 = ML$, where L and M are the differential operators defined in formulae (31), we immediately see that $\epsilon L\phi^{(2)} = LH_2\phi^{(2)} = LML\phi^{(2)} = H_1 L\phi^{(2)}$, which means that $L\phi^{(2)}$ must be proportional to $\phi^{(1)}$, or

$$\phi^{(1)}(x) = \mathcal{C}L\phi^{(2)}(x) \ ,$$

where $\mathcal{C}$ is a constant to be determined. Let us assume now that $\phi^{(2)}(x)$ is a progressive wave, with asymptotic behaviour

$$\begin{array}{c} \lim_{x\to-\infty}\phi^{(2)}(x) = e^{ik_-x} + R^{(2)}_{L\to R}e^{-ik_-x} \ , \\ \lim_{x\to+\infty}\phi^{(2)}(x) = T^{(2)}_{L\to R}e^{ik_+x} \ , \end{array}$$

where $k_\pm = \sqrt{2m\left(\epsilon - U_2\left(\pm\infty\right)\right)} = \sqrt{E^2 - m^2 - P_\pm^2}$ are the asymptotic momenta. Thus, we have

$$\begin{aligned} \lim_{x\to-\infty}\phi^{(1)}(x) &= \mathcal{C}\lim_{x\to-\infty}L\phi^{(2)}(x) = \frac{\mathcal{C}}{\sqrt{2m}}\left(\frac{d}{dx} + P_-\right)\left(e^{ik_-x} + R^{(2)}_{L\to R}e^{-ik_-x}\right) \\ &= \frac{\mathcal{C}}{\sqrt{2m}}\left[\left(ik_- + P_-\right)e^{ik_-x} + \left(-ik_- + P_-\right)R^{(2)}_{L\to R}e^{-ik_-x}\right] \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to+\infty}\phi^{(1)}(x) &= \mathcal{C}\lim_{x\to+\infty}L\phi^{(2)}(x) = \frac{\mathcal{C}}{\sqrt{2m}}\left(\frac{d}{dx} + P_+\right)T^{(2)}_{L\to R}e^{ik_+x} \\ &= \frac{\mathcal{C}}{\sqrt{2m}}\left(ik_+ + P_+\right)T^{(2)}_{L\to R}e^{ik_+x} \ . \end{aligned}$$

On the other hand, we know that

$$\begin{array}{c} \lim_{x\to-\infty}\phi^{(1)}(x) = e^{ik_-x} + R^{(1)}_{L\to R}e^{-ik_-x} \ , \\ \lim_{x\to+\infty}\phi^{(1)}(x) = T^{(1)}_{L\to R}e^{ik_+x} \ . \end{array}$$

Hence we obtain formulae (33) of the text

$$\begin{array}{r}\frac{\mathcal{C}}{\sqrt{2m}}=\frac{1}{ik_{-}+P_{-}}\\ R_{L\to R}^{(1)}=\frac{-ik_{-}+P_{-}}{ik_{-}+P_{-}}R_{L\to R}^{(2)}\\ T_{L\to R}^{(1)}=\frac{ik_{+}+P_{+}}{ik_{-}+P_{-}}T_{L\to R}^{(2)}\end{array}$$

One proceeds in a similar way for regressive waves, $\phi^{(1)}(x)=\mathcal{C}'L\phi^{(2)}(x)$, with asymptotic behaviour

$$\begin{array}{r}\lim_{x\to-\infty}\phi^{(j)}(x)=T_{R\to L}^{(j)}e^{-ik_{-}x}\ ,\\ \lim_{x\to+\infty}\phi^{(j)}(x)=e^{-ik_{+}x}+R_{R\to L}^{(j)}e^{ik_{+}x}\ ,\end{array}\quad (j=1,2)$$

with the result

$$\begin{array}{r}\frac{\mathcal{C}'}{\sqrt{2m}}=\frac{1}{-ik_{+}+P_{+}}\\ R_{R\to L}^{(1)}=\frac{ik_{+}+P_{+}}{-ik_{+}+P_{+}}R_{R\to L}^{(2)}\\ T_{R\to L}^{(1)}=\frac{-ik_{-}+P_{-}}{-ik_{+}+P_{+}}T_{R\to L}^{(2)}\end{array}\ ,$$

corrresponding to formulae (34) of the text.

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